# On the canonical transformation theorem of Currie and Saletan



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COMMENT

# On the canonical transformation theorem of Currie and Saletan

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**Abstract.** The distinction between canonical and canonoid transformations, introduced by Currie and Saletan, is here emphasised through an example, which also shows that the initial condition of their theorem was quite strict. A step forward is taken by reducing the requirement that all quadratic Hamiltonians be canonoid to the requirement that a finite number of them be canonoid.

It is often advantageous in a physical problem to find a coordinate transformation which simplifies Hamilton's equations whilst preserving their canonical form. Such a transformation is said to be *canonoid* with respect to that particular Hamiltonian.

It is commonly believed that it is then necessarily canonical, i.e. that this property will hold as well for any other Hamiltonian; but we must be careful. In fact, whether we impose this condition on a particular system or a general one, the matter at issue may be quite different.

The distinction becomes evident in the following example. Let us consider the transformation

$$q' = q\,e^{2t} \qquad p' = p^3.$$

The Hamiltonian

$$H = cqp$$

leads to

$$\dot{q} = \frac{\partial H}{\partial p} = cq \qquad \dot{p} = -\frac{\partial H}{\partial q} = -cp$$

$$\dot{q}' = \dot{q}\,e^{2t} + 2q\,e^{2t} = (c+2)q' \equiv \frac{\partial H'}{\partial p'}$$

$$\dot{p}' = 3p^2\dot{p} = -3cp' \equiv -\frac{\partial H'}{\partial q'}.$$

The condition of integrability for the existence of $H'$ is

$$\frac{\partial^2 H'}{\partial q'\partial p'} = \frac{\partial^2 H'}{\partial p'\partial q'} \Leftrightarrow c = 1.$$

Hence this transformation is canonoid for $H = qp$ but not canonical.

Furthermore, the example encountered illustrates a surprising fact: even in the case that a transformation is canonoid for one or several Hamiltonians, it is not necessarily canonoid for a linear combination of them.

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The distinction between canonoid and canonical was first pointed out by Currie and Saletan (1972), who also demonstrated that a transformation is canonical if it is canonoid for all quadratic Hamiltonians:

$$H = C + c_\alpha \xi_\alpha + \tfrac{1}{2}\omega_{\alpha\beta}\xi_\alpha\xi_\beta$$

where the $\xi$ are the generalised coordinates

$$\xi_\alpha = \begin{cases} q_\alpha & \alpha = 1, \ldots, n \\ p_{\alpha-n} & \alpha = n+1, \ldots, 2n. \end{cases}$$

The worth of Currie and Saletan's theorem is that it permits one to extend the property of being canonoid from a certain class of Hamiltonians to all others. Nevertheless, it starts from an *infinite* number of Hamiltonians and the condition required turns out to be still very strict, even though it deals with a linear combination of $2n^2+3n+1$ basic Hamiltonians (remember the conclusion from the example above).

The main purpose of the present contribution is to enhance Currie and Saletan's theorem by reducing the initial condition to a *finite* set of $5n$ Hamiltonians. A second aim is to offer a self-contained proof, without the reference to Schur's lemma required in the original work. The new statement could be set up as follows.

*Theorem.* An invertible transformation $\eta_\alpha = \eta_\alpha(\xi, t)$ is canonical if it is canonoid with respect to the following Hamiltonians:

(i) $H = 0$

(ii) $H = \xi_i, H = \xi_i^2$ $\qquad i = 1, \ldots, 2n$

(iii) $H = \xi_i\xi_{i+1}$ $\qquad i = 1, \ldots, n-1.$

*Preliminaries for the proof.* Let us define the $2n \times 2n$ matrix

$$\Gamma = \begin{pmatrix} 0_n & \mathrm{I}_n \\ -\mathrm{I}_n & 0_n \end{pmatrix}$$

which is unimodular, antisymmetric:

$$\gamma_{\alpha\beta} = -\gamma_{\beta\alpha}$$

and orthogonal:

$$\gamma_{\alpha\sigma}\gamma_{\beta\sigma} = \delta_{\alpha\beta}. \tag{1}$$

(In accordance with Einstein's convention there is an implicit sum over all repeated *Greek* indices.)

This allows us to write Hamilton's canonical equations as

$$\dot{\xi}_\alpha = \gamma_{\alpha\beta}\frac{\partial H}{\partial \xi_\beta}$$

and the Poisson bracket within the $\xi$ coordinates as

$$[F, G]^\xi = \frac{\partial F}{\partial \xi_\alpha}\gamma_{\alpha\beta}\frac{\partial G}{\partial \xi_\beta}.$$

For an invertible transformation on phase space $\eta_\alpha = \eta_\alpha(\xi, t)$, any function can be expressed as depending on the new variables:

$$A(\xi, t) = \tilde{A}(\eta, t).$$

Accordingly, the respective Poisson bracket becomes

$$[F, G]^\eta = \frac{\partial \tilde{F}}{\partial \eta_\alpha} \gamma_{\alpha\beta} \frac{\partial \tilde{G}}{\partial \eta_\beta} = \frac{\partial F}{\partial \xi_\alpha} \mu_{\alpha\beta} \frac{\partial G}{\partial \xi_\beta} \tag{2}$$

where we have defined the $\mu$ as the functions

$$\mu_{\alpha\beta}(\xi, t) \equiv [\xi_\alpha, \xi_\beta]^\eta = -\mu_{\beta\alpha}. \tag{3}$$

It can be shown that

$$\dot{\tilde{A}} = \dot{A} \equiv \frac{\partial \tilde{A}}{\partial \eta_\alpha} \dot{\tilde{\eta}}_\alpha + \frac{\partial \tilde{A}}{\partial t}.$$

These $\dot{\eta}$ are of course given by

$$\dot{\eta}_\alpha = \frac{\partial \eta_\alpha}{\partial \xi_\beta} \dot{\xi}_\beta + \frac{\partial \eta_\alpha}{\partial t} = \frac{\partial \eta_\alpha}{\partial \xi_\beta} \gamma_{\beta\sigma} \frac{\partial H}{\partial \xi_\sigma} + \frac{\partial \eta_\alpha}{\partial t}. \tag{4}$$

From a *lemma* presented in Currie and Saletan (1972), this transformation will be canonoid for $H$, i.e. there will exist a new Hamiltonian $H'(\eta, t)$ such that

$$\dot{\eta}_\alpha = \gamma_{\alpha\beta} \frac{\partial H'}{\partial \eta_\beta} \tag{5}$$

if and only if it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t} [F, G]^\eta = [\dot{F}, G]^\eta + [F, \dot{G}]^\eta \tag{6}$$

for every pair of differentiable functions $\tilde{F}(\eta, t)$ and $\tilde{G}(\eta, t)$.

*Proof of the theorem.* On account of the definition (3), every function $\mu_{\alpha\beta}(\xi, t)$ is independent of $H$, although its total time derivative, given by

$$\frac{\mathrm{d}\mu_{\alpha\beta}}{\mathrm{d}t} = \frac{\partial \mu_{\alpha\beta}}{\partial \xi_\rho} \gamma_{\rho\sigma} \frac{\partial H}{\partial \xi_\sigma} + \frac{\partial \mu_{\alpha\beta}}{\partial t}$$

is expected to depend on the actual Hamiltonian.

Nevertheless, as shown in Currie and Saletan (1972), conditions (i) and (ii) of the theorem imply, respectively, that

$$\frac{\partial \mu_{\alpha\beta}}{\partial t} = 0 \qquad \text{and} \qquad \frac{\partial \mu_{\alpha\beta}}{\partial \xi_\rho} = 0.$$

Therefore,

$$\frac{\mathrm{d}\mu_{\alpha\beta}}{\mathrm{d}t} = 0 \tag{7}$$

*for all Hamiltonians.*

In particular, we take $H = \xi_i^2$, $i = 1, \ldots, 2n$. The motion is

$$\dot{\xi}_\alpha = \gamma_{\alpha\beta} \frac{\partial \xi_i^2}{\partial \xi_\beta} = 2\gamma_{\alpha i} \xi_i.$$

(There is no sum on *Latin* indices.)

When the lemma is applied, property (6) turns into

$$0=\frac{\mathrm{d}\mu_{\alpha\beta}}{\mathrm{d}t}=2\gamma_{\alpha i}\mu_{i\beta}+2\gamma_{\beta i}\mu_{\alpha i}$$

which is equivalent to

$$\gamma_{i\alpha}\mu_{i\beta}=\gamma_{i\beta}\mu_{i\alpha}. \tag{8}$$

We define $2n$ constant $c$ by

$$c_i\equiv\mu_{i\sigma}\gamma_{i\sigma}=\mu_{\sigma i}\gamma_{\sigma i}.$$

After multiplying (8) by $\gamma_{i\beta}$, summing over $\beta$ and remembering (1), we obtain

$$\gamma_{\alpha i}c_i=\mu_{\alpha i}\qquad\forall i. \tag{9}$$

In a similar fashion, for the Hamiltonian $H=\xi_i\xi_{i+1}$, $i=1,\ldots,n-1$, we have

$$\dot{\xi}_\alpha=\gamma_{\alpha\beta}\frac{\partial(\xi_i\xi_{i+1})}{\partial\xi_\beta}=\gamma_{\alpha i}\xi_{i+1}+\gamma_{\alpha i+1}\xi_i$$

and property (6) has now the form

$$0=\frac{\mathrm{d}\mu_{\alpha\beta}}{\mathrm{d}t}=\gamma_{\alpha i}\mu_{i+1\beta}+\gamma_{\alpha i+1}\mu_{i\beta}+\gamma_{\beta i}\mu_{\alpha i+1}+\gamma_{\beta i+1}\mu_{\alpha i}.$$

If we multiply it by $\gamma_{i\alpha}\gamma_{i+1\beta}$ and proceed as before, we arrive at

$$c_{i+1}=c_i\qquad i=1,\ldots,n-1.$$

From (9) we observe that

$$\gamma_{ij}c_i=\mu_{ij}=-\mu_{ji}=-\gamma_{ji}c_j=\gamma_{ij}c_j.$$

For $j=i+n$, $\gamma_{ij}=1$ and so

$$c_i=c_{i+n}.$$

This means that

$$c_1=c_2=\ldots=c_{2n}\equiv z$$

which, introduced into (9), works up to

$$z\gamma_{ij}=\mu_{ij}=\frac{\partial\tilde{\xi}_i}{\partial\eta_\alpha}\gamma_{\alpha\beta}\frac{\partial\tilde{\xi}_j}{\partial\eta_\beta}.$$

We prove that $z$ is not null by taking determinants:

$$z=\left|\frac{\partial\tilde{\xi}_\alpha}{\partial\eta_\beta}\right|^2=\left|\frac{\partial\eta_\alpha}{\partial\xi_\beta}\right|^{-2}>0.$$

Henceforth, from (2),

$$[F,G]^\eta=z[F,G]^\xi.$$

This is a well known condition for a transformation to be canonical. In fact, for *any* given Hamiltonian $H$

$$\frac{\mathrm{d}}{\mathrm{d}t}[F,G]^\eta=z([\dot{F},G]^\xi+[F,\dot{G}]^\xi)=[\dot{F},G]^\eta+[F,\dot{G}]^\eta$$

and the lemma can now be applied inversely.

*Final remarks.* From (4) and (5) it is straightforward to show that a transformation canonoid for some Hamiltonians is also canonoid for every linear combination of them if it is canonoid for $H=0$.

It is also not difficult to prove that if

$$\frac{\partial \eta_\alpha}{\partial t} = \text{constant}$$

(as is the case for a time-independent transformation), then $\eta_\alpha = \eta_\alpha(\xi, t)$ is canonoid for $H=0$. And if the partial derivatives

$$\frac{\partial \eta_\alpha}{\partial \xi_\beta}$$

are constant as well, it will also be canonoid for every $H = \xi_i$.

For the application of the theorem, these properties may be a valuable help in many of the most usual cases.

## Reference

Currie D G and Saletan E J 1972 *Nuovo Cimento* B 9 143